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1985 J. Phys. A: Math. Gen. 18 49

(http://iopscience.iop.org/0305-4470/18/1/016)

J. Phys. A: Math. Gen. **18** (1985) 49-60. Printed in Great Britain

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Received 19 June 1984

Abstract. This paper describes a Monte Carlo study of the 2D kinetic Ising model. The long-time behaviour of various time-delayed correlation functions is investigated by renormalisation group methods. This long-time behaviour can be described in terms of a simple exponential relaxation of the magnetisation which enables a reliable estimate of the dynamical critical exponent, z, to be made.

1. Introduction

A substantial amount of recent work in critical dynamics has been devoted to the need to develop effective real space dynamic renormalisation group methods (Mazenko and Valls 1982 and references therein). The two-dimensional kinetic Ising model has provided a suitable and demanding testing ground, with the determination of the dynamic exponent, z, being the main focus of attention (Mazenko and Valls 1981). Methods which give good results for static critical phenomena often behave very erratically when applied to dynamic properties; this has been attributed to the narrowness of the asymptotic dynamic critical region relative to the asymptotic static critical region. A promising new approach involves the combination of Monte Carlo computer simulations at the critical point T_c, with real space renormalisation. The idea of Monte Carlo renormalisation was first applied to static critical phenomena, and has been very successful, producing accurate and reliable results (Swendsen 1982 and references therein). Several variants of the method have evolved and, in particular, a scheme suggested by Wilson (unpublished) has been extended by several authors (Tobochnik *et al* 1981, Katz *et al* 1982) to deal with dynamic critical phenomena.

The principal limitation of standard Monte Carlo techniques is the finite-size effect: this is most acute in the critical region and prevents the accurate determination of critical properties. The Wilson scheme confronts the finite-size problem by using two simulations on lattices of different sizes. Starting with two lattices, which differ in size by a factor b, a standard Monte Carlo simulation is employed to generate a sequence of spin configurations characteristic of the starting Hamiltonian at T_c on a finite lattice. From these configurations a sequence of block-spin configurations is created by a real space renormalisation transformation, typically by 'majority-rule blocking', which reduces the linear dimensions of the system by a factor b. This sequence of block-spin configurations is characteristic of a renormalised Hamiltonian, in the sense that any correlation function calculated from these configurations is an average over the block-spin configurations weighted by the Boltzmann factor appropriate to that renormalised

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0305-4470/85/010049+12$02.25

Hamiltonian. The procedure may be iterated, and since the simulations start on the critical surface further iterations are expected to keep the renormalised Hamiltonians on the critical surface and to move them towards a fixed point. If m blockings are performed on the smaller lattice and $m+1$ on the larger lattice the two block-spin lattices produced will have the same number of block spins, both systems will show finite-size effects but these effects should be the same for both. If the Hamiltonian derived from m iterations on the smaller lattice equals that derived from $m+1$ iterations on the larger lattice, then the correlation functions based on these Hamiltonians will be equal. According to dynamic scaling if the length is scaled by a factor b then time scales by a factor b^z, which defines the dynamic critical exponent z. This means that if a time-dependent correlation function, $f(t)$ say, is measured on the smaller lattice after m blockings and the same correlation function, $f(t')$, is measured on the larger lattice after $m+1$ blockings then they are expected to 'match' at times t and t', where $t'/t = b^z$ (where time is measured in units defined on the 'bare' lattices). By matching various time correlation functions z can be determined.

In this paper we extend the earlier work of Tobochnik *et al* (1981) and of Katz *et al* (1982) on the 2D kinetic Ising model, employing a novel updating scheme and exploring both larger lattices and longer time delays in the correlation functions. As in the previous studies we investigate the time dependence of the spin and block-spin autocorrelation functions and also the time-delayed spin and block-spin nearest-neighbour correlation functions. Our results, giving $z = 2.13\,(3)$ ((3) indicating the estimated error in the final digit), suggest that the earlier work overestimated the value of z by a few per cent; indeed Katz *et al* encountered some problems in obtaining a unique value for z from their data. More specifically, they attempted to match correlation functions at a given time delay, t, by fitting their data to a simple exponential decay (the expected long-time behaviour), unfortunately the value obtained for z was dependent on the value of t (z decreasing with increasing t); an indication, perhaps, that the time delays were not long enough for the correlation functions to attain their true asymptotic behaviour. In our own simulations we find that a simple exponential decay fits the long-time behaviour extremely well and, furthermore, the same decay times are found for the local spin and block-spin time-delayed correlation functions and also for the relaxation of the magnetisation. Since the magnetisation is the only slow mode in the model and is 'captured' by the long-time behaviour of all the correlation functions, as indicated by the decay times found, we can be confident that our data lies in the required asymptotic long-time regime. The renormalisation or blocking transformations serve to enhance the amplitude of this mode in the block-spin correlation functions and the scaling behaviour of this amplitude lends further credence to the claim that we are seeing the true critical behaviour.

The structure of the paper is as follows. In § 2 we define the model and briefly discuss some of its properties. In § 3 we present the basic calculational procedures and the results of our simulations. Finally in § 4 we summarise our results and add a few concluding remarks.

2. The kinetic Ising model

The model we will consider is the 2D kinetic Ising or Glauber model which belongs to the dynamic universality class where the non-conserved order parameter is the only slow mode ('model A' in the usual terminology). The model consists of N sites

belonging to an $L \times L$ square lattice with Ising spins, σ_i, located at each site i, the spin coordinate, σ_i, assumes the values ± 1. A configuration of the model is completely described by specifying the state of all the spins $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$ and is denoted by $\{\sigma\}$. We only consider nearest-neighbour interactions, so the Hamiltonian of the system, including the inverse temperature factor $(-\beta)$, is

$$\mathcal{H}(\{\sigma\}) = K \sum_{\langle ij \rangle} \sigma_i \sigma_j \tag{2.1}$$

where the summation extends over all nearest-neighbour pairs. The dynamical evolution of the model is controlled by a Master equation for the probability function $P(\{\sigma\}; t)$ which gives the probability that a configuration $\{\sigma\}$ occurs at time t given some initial probability function at time $t = 0$. The Master equation is given by

$$\frac{\partial}{\partial t} P(\{\sigma\}; t) = \sum_{\{\sigma\}'} (P(\{\sigma\}'; t) W(\{\sigma\}' \to \{\sigma\}) - P(\{\sigma\}; t) W(\{\sigma\} \to \{\sigma\}')) \tag{2.2}$$

where $W(\{\sigma\} \to \{\sigma\}')$ is the transition probability per unit time for a configuration to change from $\{\sigma\}$ to $\{\sigma\}'$. We are free to choose the form of the transition probabilities W, subject to the requirements of detailed balance and ergodicity which ensure that the probability function $P(\{\sigma\}; t)$ evolves towards the correct equilibrium probability function $P_{\text{eq}}(\{\sigma\})$. The dynamic exponent z, being a universal quantity, should be independent of the choice of the transition probabilities.

Following Yalabik and Gunton (1979) we can rewrite the Master equation in matrix form,

$$\frac{\partial P}{\partial t} = LP \tag{2.3}$$

where the 'Liouville operator' is given by

$$\begin{aligned} L(\{\sigma\} \to \{\sigma\}') &= W(\{\sigma\} \to \{\sigma\}'), && \{\sigma\} \neq \{\sigma\}' \\ &= -\sum_{\{\sigma\}''} W(\{\sigma\} \to \{\sigma\}''), && \{\sigma\} = \{\sigma\}'. \end{aligned}$$

The formal solution can then be written as

$$P(\{\sigma\}; t) = \exp(tL) P(\{\sigma\}; 0) \tag{2.4}$$

or

$$P(\{\sigma\}; t) = P_{\text{eq}}(\{\sigma\}) + a_1 \, e^{\lambda_1 t} P_1(\{\sigma\}) + a_2 \, e^{\lambda_2 t} P_2(\{\sigma\}) + \ldots \tag{2.5}$$

where $P_i(\{\sigma\})$ and λ_i denote the ith eigenvector and eigenvalue of L respectively, and the a_i are time-independent expansion coefficients. Detailed balance guarantees that $P_{\text{eq}}(\{\sigma\})$ is an eigenvector of L with eigenvalue zero. For a finite system all other eigenvalues of L are negative so that $P(\{\sigma\}; t)$ eventually decays to equilibrium. As L is invariant under a global inversion of the spins the eigenfunctions can be divided into odd and even functions. From equation (2.5) it is clear that the long-time behaviour of any odd or even correlation function will be determined by the smallest odd or even eigenvalue respectively. In principal we might expect different dynamic exponents for thermal (even) and magnetic (odd) correlation functions but it is believed that there is only one relevant time scale in critical dynamics just as there is only one length scale, the correlation length, in the statics. So for the rest of this paper we will only consider odd correlation functions and we will be concerned with calculating

$|\lambda_1^0|$, the smallest odd eigenvalue. This eigenvalue determines the critical relaxation time and for an infinite system we expect,

$$|\lambda_1^0| \sim [(T - T_c)/T_c]^{\nu z} \tag{2.6}$$

and for a finite system at T_c finite size scaling predicts

$$|\lambda_1^0(L)|^{-1} \sim L^z \tag{2.7}$$

where ν is the correlation length exponent.

3. Method and results

The basic ideas behind the dynamic Monte Carlo renormalisation group method have already been briefly outlined in the introduction; for a more detailed discussion see Tobochnik *et al* (1981). As in the previous studies we consider the following types of correlation function

$$\begin{aligned} \Gamma(L, m; t) &= (N^{(m)})^{-1} \left\langle \sum_i \sigma_i^{(m)}(0)\sigma_i^{(m)}(t) \right\rangle \\ E(L, m; t) &= (N^{(m)})^{-1} \left\langle \sum_{\langle ij \rangle} \sigma_j^{(m)}(0)\sigma_i^{(m)}(t) \right\rangle \end{aligned} \tag{3.1}$$

where L is the lattice size, m is the number of renormalisation transformations performed, $\sigma_i^{(m)}$ is a block spin taking on the values ± 1, $N^{(m)}$ is the number of block spins on the renormalised lattice, and $\langle ij \rangle$ denotes nearest-neighbour pairs. We estimate these correlation functions from Monte Carlo simulations on lattices of size $L = 8$, 16, 32 and 64. The simulations provide sequences of spin configurations characteristic of the Hamiltonian (2.1) at critically, T_c. A 'majority rule' renormalisation transformation is performed on each configuration to obtain sequences of block-spin configurations. The 'majority rule' transformation involves dividing the lattice into 2×2 blocks and assigning block spins of $+1$ or -1 according to the sign of the sum of the spins in each block; in the event of a zero block sum the block spin is randomly assigned. The transformation reduces the linear dimensions of the system by a factor $b = 2$. This procedure can be usefully repeated until the size of the renormalised lattice reaches a limit determined by the range of the interactions in the fixed point Hamiltonian; further iterations beyond this limit will be strongly affected by finite-size effects. Having obtained sequences of spin and block-spin configurations, we can estimate the correlation functions by simply averaging over the respective configurations.

The underlying simulation uses a novel updating scheme to simulate the behaviour of the Master equation (2.2), abandoning the conventional single-spin-flip schemes for a multi-spin-flip scheme. This dictates a discrete 'timestep by timestep' approach to the dynamics with many spin flips taking place within a single time step. Each update can be regarded as being equivalent to N single-spin-flip trials, where all the spins on one sublattice are on trial, with the other sublattice passive, before the roles of the sublattices are interchanged. The N spin-flip trials constitute one update and define a unit time step of one Monte Carlo Step per spin (MCS/spin). Although there are dynamical differences between this scheme and the usual random single-spin-flip schemes, arising from the sequential order in which the spins are sampled, we do not

expect the universal critical behaviour to be affected by the choice of update scheme. A more detailed discussion of update schemes and dynamics giving support to the above claim can be found in Williams (1984). Within this update scheme we still have the freedom to choose the form of the transition probabilities for the spin-flip trials and we use both the Metropolis and the Glauber schemes, providing a partial check on the universal nature of the results. This rather unusual choice of updating, from the dynamical point of view, has been made to exploit the parallel achitecture of the array processor (ICL DAP) used for the simulations; all the spin-flip trials on a given sublattice are processed simultaneously allowing an enormous saving in computer time and enabling long simulations to be undertaken.

Some details of the simulations are given in table 1 for the longest runs, these used hot starts and Metropolis flipping probabilities. These runs were repeated with cold starts, and additional shorter runs, using both Metropolis or Glauber updating and hot or cold starts, were also undertaken. The results for $\Gamma(t)$ and $E(t)$ for the 32×32 and 16×16 lattices, taken from the runs given in table 1, are shown in table 2 and table 3 respectively. The numbers in parentheses give the approximate statistical uncertainty in the last digit(s) as estimated either from the spread of values obtained from blocks of data evaluated separately (for the larger lattices) or from the sample to sample spread of values (for the smaller lattices). In table 4 we present results for the static quantity $E(0)$ for all four lattice sizes (same runs as table 1), together with data from Tobochnik *et al* and Katz *et al.*

Table 1. Simulation details.

Lattice size (L)	64	32	16	8
Number of samples	1	4	16	64
Length of run for each sample†	512	320	256	128
Steps discarded for equilibration†	20	10	10	2

† Units of 10^4 Monte Carlo Steps/spin.

Since the simulations are performed at criticality the renormalised Hamiltonians are expected to approach a fixed point and we expect the correlation functions to match, for sufficiently large m, according to

$$\begin{aligned}\Gamma(L, m; t) &= \Gamma(bL, m+1; b^z t)\\ E(L, m; t) &= E(bl, m+1; b^z t).\end{aligned} \tag{3.2}$$

Inspection of the results in table 4 confirms this behaviour for the static quantities $E(L, m; 0)$ and $E(bL, m+1; 0)$, with matching to within statistical error for $m>2$. This fairly rapid convergence towards the fixed point is typical for the 2D Ising model and 'majority rule' blocking; see for instance Swendsen (1982). Our results for the static quantities are in good agreement with those of Katz *et al* but do not agree, at least for the renormalised quantities, with those obtained by Tobochnik *et al.* This discrepancy in the renormalised correlation functions may arise from the choice of tie breaker used when the sum of the spins in a block is zero; however, while the correlation functions seem sensitive to this choice the universal quantities are not as indicated by the agreement in the estimates of z by both Tobochnik *et al* and Katz *et al.*

Table 2. Results for $\Gamma(L, m; t)$.

t	m	$L=32$	t	m	$L=16$
80	0	0.4224 (6)	20		
	1	0.4979 (7)		0	0.5008 (4)
	2	0.5906 (9)		1	0.5897 (5)
	3	0.6991 (10)		2	0.6968 (5)
160	0	0.3951 (8)	40		
	1	0.4658 (10)		0	0.4662 (5)
	2	0.5526 (11)		1	0.5490 (5)
	3	0.6543 (13)		2	0.6485 (6)
240	0	0.3711 (10)	60		
	1	0.4375 (11)		0	0.4347 (5)
	2	0.5188 (13)		1	0.5120 (6)
	3	0.6141 (16)		2	0.6050 (7)
320	0	0.3483 (11)	80		
	1	0.4105 (13)		0	0.4054 (6)
	2	0.4869 (16)		1	0.4775 (7)
	3	0.5764 (19)		2	0.5643 (8)
400	0	0.3269 (12)	100		
	1	0.3854 (15)		0	0.3782 (7)
	2	0.4570 (17)		1	0.4453 (9)
	3	0.5410 (20)		2	0.5262 (10)
480	0	0.3069 (14)	120		
	1	0.3618 (16)		0	0.3530 (8)
	2	0.4291 (19)		1	0.4159 (9)
	3	0.5081 (23)		2	0.4914 (11)
560	0	0.2882 (16)	140		
	1	0.3396 (18)		0	0.3300 (8)
	2	0.4028 (22)		1	0.3887 (9)
	3	0.4770 (26)		2	0.4595 (13)

In order to match the dynamic correlation functions we attempt to fit the data to a simple exponential decay, e.g.,

$$\ln \Gamma(L, m; t) = a(L, m) - \lambda(L, m)t. \tag{3.3}$$

Typical results are shown in figure 1 and values for $a(L, m)$ are given in table 5. We find that $\lambda(L, m)$ is independent of m and values of $\lambda(L)$ are given in table 6. We also get similar results for $E(L, m; t)$, obtaining the same values for $\lambda(L)$ within statistical errors as those obtained from $\Gamma(L, m; t)$; indeed tables 2 and 3 show that for large t (in this case all the data points given!) $E(L, m; t) - 2\Gamma(L, m; t)$. This suggests that we are seeing the critical decay mode characterised by $\lambda_1^0(L)$, the smallest odd eigenvalue of the Liouville operator. The long time behaviour of the correlation functions reflects the decay of the residual magnetisation which is present on a finite lattice even at T_c. A typical record of the magnetisation fluctuations is shown in figure 2: we see that the average magnitude of the magnetisation, $M_0 = \langle|\Sigma\sigma_i|\rangle$, is quite large and for most of the time the magnetisation fluctuates about either $+M_0$ or $-M_0$ with sudden switches from one sign of the magnetisation to the other. This behaviour can be understood in terms of the very strong increase in ordering just below T_c and the enhancement of this ordering by the imposed periodic boundary conditions. The dominant mechanism for the magnetisation reversal involves the sweeping of a pair

Table 3. Results for $E(L, m; t)$.

t	m	$L=32$	t	m	$L=16$
0	0	1.4334 (2)	0		
	1	1.4291 (4)		0	1.4532 (3)
	2	1.4651 (7)		1	1.4704 (7)
	3	1.5466 (12)		2	1.5479 (10)
80	0	0.8449 (13)	20		
	1	0.9959 (15)		0	1.0015 (7)
	2	1.1809 (19)		1	1.1793 (13)
	3	1.3964 (22)		2	1.3927 (15)
160	0	0.7903 (16)	40		
	1	0.9317 (19)		0	0.9324 (9)
	2	1.1052 (23)		1	1.0982 (10)
	3	1.3085 (27)		2	1.2972 (11)
240	0	0.7422 (18)	60		
	1	0.8749 (22)		0	0.8694 (10)
	2	1.0375 (27)		1	1.0239 (12)
	3	1.2280 (32)		2	1.2098 (14)
320	0	0.6965 (22)	80		
	1	0.8209 (27)		0	0.8109 (11)
	2	0.9737 (33)		1	0.9550 (14)
	3	1.1526 (38)		2	1.1288 (15)
400	0	0.6538 (25)	100		
	1	0.7708 (30)		0	0.7562 (14)
	2	0.9140 (35)		1	0.8905 (16)
	3	1.0819 (42)		2	1.0524 (19)
480	0	0.6139 (28)	120		
	1	0.7236 (33)		0	0.7061 (16)
	2	0.8583 (39)		1	0.8316 (19)
	3	1.0163 (46)		2	0.9827 (21)
560	0	0.5763 (31)	140		
	1	0.6793 (37)		0	0.6600 (16)
	2	0.8056 (44)		1	0.7777 (18)
	3	0.9538 (53)		2	0.9191 (22)

Table 4. Results for $E(L, m; 0)$.

m	$L=64$	m	$L=32$	m	$L=16$	m	$L=8$
0	1.4236 (5)						
1	1.4084 (10)	0	1.4334 (2)				
2	1.4240 (18)	1	1.4291 (4)	0	1.4532 (3)		
3	1.4653 (32)	2	1.4651 (7)	1	1.4704 (7)	0	1.4918 (2)
4	1.5462 (49)	3	1.5466 (12)	2	1.5479 (10)	1	1.5514 (4)
Data from		0	1.4336 (3)				
Katz *et al*		1	1.4287 (6)	0	1.4519 (9)		
		2	1.4642 (12)	1	1.4680 (18)		
		3	1.5442 (22)	2	1.5439 (32)		
Data from				0	1.4529 (11)		
Tobochnik *et al*				1	1.4585 (37)	0	1.4909 (6)
				2	1.6059 (46)	1	1.6039 (12)

Table 5. Results for $-a(L, m)$.

m	$L=64$	m	$L=32$	m	$L=16$	m	$L=8$
0	0.9731 (75)						
1	0.8086 (76)	0	0.7989 (17)				
2	0.6370 (75)	1	0.6343 (17)	0	0.6238 (6)		
3	0.4637 (74)	2	0.4636 (17)	1	0.4604 (6)	0	0.4463 (4)
4	0.2945 (67)	3	0.2950 (17)	2	0.2934 (6)	1	0.2867 (4)

Table 6. Results for $\lambda(L)$.

	$L=64$	$L=32$	$L=16$	$L=8$
$\lambda(L)$	1.82 (8)	7.99 (9)	34.81 (10)	139.2 (2)

Units of 10^{-4} (MCS/spin)$^{-1}$.

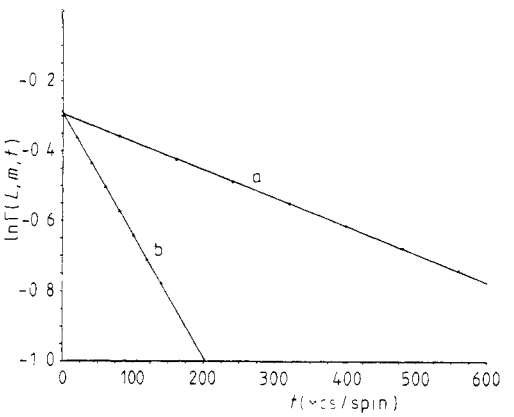

Figure 1. A plot of Log $\Gamma(L, m; t)$ against t measured in MCS/spin. Errors are comparable to the symbol size. Line a; $L=32$, $m=3$; line b; $L=16$, $m=2$.

of interfaces across the lattice; the interfaces arise from the coalescing of large droplets of overturned spins and once formed move apart, eventually meeting and annihilating on the 'far side' of the lattice. The characteristic time for this reversal is reflected in $\lambda_1^0(L)$.

Further support for this picture comes from considering the magnetisation-magnetisation autocorrelation function defined by

$$\Gamma_M(L; t) = \langle M(t)M(0)\rangle / \langle M(0)M(0)\rangle \tag{3.4}$$

where

$$M(t) = \sum_i \sigma_i(t).$$

We find that the long-time behaviour of this quantity is again characterised by the same exponential decay

$$\ln \Gamma_M(L; t) = C(L) - \Lambda_M(L)t. \tag{3.5}$$

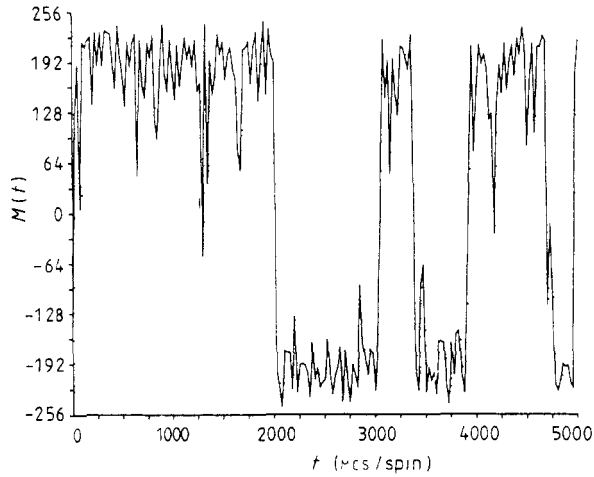

Figure 2. A record of the magnetisation fluctuations on a 16×16 lattice.

Results for $\lambda(L)$ and $\Lambda_M(L)$ are given in table 7. These results were taken from different runs to those used in the blocking analysis (for computational reasons): the data for $L = 16$ was taken from 16 samples running for 10 MCS/spin (for each sample) and for $L = 32$ it was taken from four samples running for 10 MCS/spin (for each sample).

Table 7. Results for $\lambda(L)$ and $\Lambda_M(L)$.

	$L = 32$	$L = 16$
$\lambda(L)$	8.16 (13)	35.07 (12)
$\Lambda_M(L)$	8.14 (10)	34.96 (10)

Units of 10^{-4} (MCS/spin)$^{-1}$.

The final piece of evidence in favour of this picture comes from the way $a(L, m)$ scales with m. Each renormalisation transformation reduces the dimensions of the lattice by a scale factor b, and close to the fixed point of the transformation the block-spin susceptibility should scale according to

$$\chi^{(m)} = \left\langle \left(\sum_i \sigma_i^{(m)} \right)^2 \right\rangle \sim (L/b^m)^{d+2-\eta}. \tag{3.6}$$

If the long-time behaviour of $\Gamma(L, m; t)$ and $E(L, m; t)$ is governed by the relaxation of the magnetisation fluctuations then the amplitude of this mode, $a(L, m)$, should show similar scaling behaviour to $\chi^{(m)}$. Table 8 gives estimates of η obtained by comparing $a(L, m)$ for m and $m+1$ and using the finite-size scaling form given in (3.6). These estimates are in excellent agreement with those obtained by Swendsen (1982) from a finite-size scaling analysis of static block-spin (majority rule) correlation functions.

To determine the dynamical critical exponent, z, we can either use (3.2) or, having identified $\lambda(L)$ with $\lambda_1^0(L)$, we can use (2.7). Estimates $z(L, L')$ from simulations on

Table 8. Estimates for η $(=2\beta/\upsilon)$.

m	$L=64$	$L=32$	$L=16$	$L=8$
1	0.237	0.237	0.236	0.230
2	0.248	0.246	0.241	
3	0.250	0.243		
4	0.244			

Exact value $\frac{1}{4}$.

$L\times L$ and $L'\times L'$ lattices are presented in table 9. The error estimates are based on making one standard deviation changes in opposite directions on the two lattices. The results suggest that we do not see the true dynamical critical behaviour on an 8×8 lattice; however from the other sizes and from additional shorter runs we get consistent results all yielding (within statistical error) the same value of abut 2.13. This value is also confirmed for the simulations employing Glauber rather than Metropolis flipping probabilities showing that universality holds as expected. Overall our best estimate is $z=2.13\,(3)$, the error estimate is based on the spread of values obtained from several runs and is somewhat subjective due to the need to attribute a weighting to account for the different lengths of run etc (the error is probably conservative). This value is somewhat lower than the previous estimates of Tobochnik *et al* and Katz *et al.* It seems likely that these earlier studies did not look at long enough time delays as indicated, for instance, by the different decay times found for $\Gamma(t)$ and $E(t)$ by Katz *et al.* The fact that, for a given lattice size, we find the same decay time characterises the long-time behaviour of all the odd correlation functions which we considered gives us confidence that we are looking at the dynamical critical behaviour of the model, and leads us to believe we have a more reliable estimate of z.

Table 9. Estimates of z.

$z(16,8)=2.00\,(1)$
$z(32,16)=2.12\,(2)$
$z(64,32)=2.13\,(8)$

4. Concluding remarks

In summary we are confident that we have found the correct long-time behaviour for a given choice of dynamics and lattice size, and that we see dynamic scaling on the larger lattices ($L=16$, 32, 64). Mazenko and Valls (1981) point out, however, that the existence of a dynamic scaling regime does not guarantee that it is the asymptotic scaling regime. The narrowness of the dynamical critical region implies the need to go to large length scales before asymptotic behaviour is observed. We cannot, of course, rule out the possibility that if we were to investigate even larger systems some new behaviour would set in.

In the light of the preceding remarks we compare our best estimate, $z=2.13\,(3)$, with the results of previous studies. As already mentioned our estimate is somewhat lower than those of Tobochnik *et al*, who obtained $z=2.22\,(13)$ (with a most probable

value 2.17), and Katz *et al*, who obtained $z \approx 2.23$; a possible explanation of this discrepancy is that it is difficult to match the correlation functions outside the long-time regime where a simple exponential decay predominates (the contribution from the next lowest eigenvalue is probably not negligible in their data). Other Monte Carlo studies using finite-size scaling on lattices of size $L \leqslant 16$ have given the following estimates: $z = 2.00\,(10)$ (Angles d'Auriac *et al* 1982) and $z = 2.10\,(10)$ (Takano 1982), both results were obtained from rather small lattices. Our results are in closest agreement with results obtained from high-temperature series expansions and field theory methods. Racz and Collins (1976) obtained the estimate, $z = 2.125\,(10)$ from a 12 term series; however, many more terms are needed to justify the reliability of this result. The field theory estimate given by Bausch *et al* (1981) is $z \approx 2.126$ which was obtained from a Padé interpolation between the two-loop result near $d = 1$ and the two-loop result near $d = 4$. This close agreement is almost certainly fortuitous but probably rules out the possibility that z has a simple value like 2.

Finally, we conclude with a few remarks about the method. The previous dynamic Monte Carlo renormalisation group studies utilised the renormalisation transformations directly by explicitly 'matching' the correlation functions and looking for fixed point behaviour. In our own simulations we have taken a different, but closely related, approach by focussing on the long-time behaviour and using finite-size scaling, which of course is derived from renormalisation group theory, to expose the critical behaviour. As a consequence of this approach the matching is done implicitly. We have used the blocking transformations in a subsidiary role, namely to confirm the onset of scaling behaviour and that we are exploring the long-time regime (the inverse decay time $\lambda(L)$ is an RG invariant); in this regime determining z by 'matching' correlation functions or by using finite-size scaling are equivalent.

In principle, however, the 'matching' method should work for shorter times as well near the fixed point, since the irrelevant short-time behaviour is 'integrated out' in the blocking. The 'matching' method requires that the effects of irrelevant variables die out rapidly upon successive renormalisations; the difficulty lies in finding a sensible way of interpreting the data when the contribution from the irrelevant effects cannot safely be assumed to be negligible. The alternative procedure of concentrating on the long-time behaviour relies on the exponential behaviour setting in 'cleanly', in other words that the smallest eigenvalue of the Liouville operator is well 'separated' from the next smallest eigenvalue. It suffers too from the difficulty that the Monte Carlo estimates of the correlation functions become less and less accurate as the time delays get longer. Nevertheless, we believe the present simulations have provided a reliable estimate of z that is consistent with the values found by other techniques. To improve on this requires either a much higher order series expansion or further theoretical improvements in the Monte Carlo renormalisation group method, particularly in understanding the shorter time behaviour. The latter path would seem to be the more worthwhile.

Acknowledgments

I would like to thank N D Mackenzie, C K Harris and, in particular, A D Bruce for helpful discussions and the SERC and the ERCC for providing computer time on an ICL Distributed Array Processor. I would also like to thank the SERC for financial support whilst this work was carried out.

References

Angles d'Auriac J C, Maynard R, and Rammal R 1982 *J. Stat. Phys.* **28** 307
Bausch R, Dohm V, Janssen H K, and Zia RKP 1981 *Phys. Rev. Lett.* **47** 1837
Katz S L, Gunton J D, and Chao Ping Liu 1982 *Phys. Rev.* B **25** 6008
Mazenko G F and Valls O T 1981 *Phys. Rev.* B **24** 1419
—— 1982 in *Real-space renormalisation* ed T W Burkhardt and J M J Van Leeuwen, (*Topics in Current Physics*) (Berlin: Springer)
Racz Z and Collins M F 1976 *Phys. Rev.* B **13** 3074
Swendsen R H 1982 in *Real-space renormalisation* ed T W Burkhardt and J M J Van Leeuwen (*Topics in Current Physics*) (Berlin: Springer)
Takano H 1982 *Prog. Theor. Phys.* **68** 493
Tobochnik J, Sarker S and Cordery R 1981 *Phys. Rev. Lett.* **46** 1417
Williams J K 1984 *J. Phys. A: Math. Gen.* to be submitted
Wilson K G Unpublished
Yalabik M C and Gunton J D 1979 *Prog. Theor. Phys.* **62** 1573